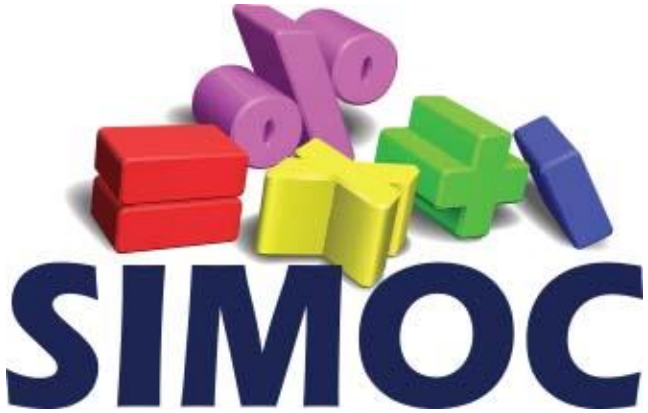
Teacher Copy

Upper Division
Grade 8, 9, 10 & 11



Example

Find the remainder when 2^{44} is divided by 5.

Solution:

Observe the following pattern: when divided by 5,

2^1 leaves a remainder of 2,

$2^2 = 4$ leaves a remainder of 4,

$2^3 = 8$ leaves a remainder of 3,

$2^4 = 16$ leaves a remainder of 1,

$2^5 = 32$ leaves a remainder of 2,

$2^6 = 64$ leaves a remainder of 4, …

$2^7 = 128$ leaves a remainder of 3,

$2^8 = 256$ leaves a remainder of 1,

This means that the remainder will repeat with a period of 4.

Since $44 = 11 \times 4$, then 2^{44}, when divided by 5, will leave a remainder of **1**

Question 1

Find the remainder when 3^{664} is divided by 7.

Solution

Observe the following pattern: when divided by 7,

3^1 leaves a remainder of 3,

$3^2 = 9$ leaves a remainder of 2,

$3^3 = 27$ leaves a remainder of 6,

$3^4 = 81$ leaves a remainder of 4,

$3^5 = 243$ leaves a remainder of 5,

$3^6 = 729$ leaves a remainder of 1,

$3^7 = 2187$ leaves a remainder of 3,

$3^8 = 6561$ leaves a remainder of 2, …

This means that the remainder will repeat with a period of 6.

Since $664 = 110 \times 6 + 4$, then 3^{664}, when divided by 7, will leave a remainder of **4**.

Question 2

Find the following sum:

$$\frac{101}{1001}+\frac{103}{1001}+\frac{105}{1001}+\cdots+\frac{1001}{1001}$$

Solution:

$1+3+5+\cdots 99=50^2=2500 \quad (99=2\times 49+1)$

$1+3+5+\cdots+1001=501^2=251{,}001 \quad (1001=2\times 500+1)$

$$\frac{101}{1001}+\frac{103}{1001}+\frac{105}{1001}+\cdots+\frac{1001}{1001}=\frac{251{,}001-2500}{1001}=\frac{248{,}501}{1001}$$

Question 3

Find the value of

$$12341235 \times 12351234 - 12341234 \times 12351235$$

Solution:

$12341235 \times 12351234 = (12341234 + 1) \times (12351235 - 1) =$

$= 12341234 \times 12351235 + 12351235 - 12341234 - 1$

$$12341235 \times 12351234 - 12341234 \times 12351235 =$$

$= 12341234 \times 12351235 + 12351235 - 12341234 - 1 - 12341234 \times 12351235$

$= 12351235 - 12341234 - 1 = 10000$

Question 4

If a and b are positive whole numbers, find the values of a and b which satisfy the equation

$$9a^2 - b^2 = 53$$

Solution:

$$9a^2 - b^2 = (3a - b)(3a + b) = 1 \times 53$$

Since $a - b < a + b$, we get $3a - b = 1$ and $3a + b = 53$

Therefore, $a = 9, b = 26$

Question 5

Find the value of

$$8+\cfrac{9}{8+\cfrac{9}{8+\cfrac{9}{8+\cfrac{9}{8+\cdots}}}}$$

Solution: Let $8+\cfrac{9}{8+\cfrac{9}{8+\cfrac{9}{8+\cfrac{9}{8+\cdots}}}}=a$

Question 6

If $x + \frac{1}{x} = 2$ and $x > 0$, find $x^3 + \frac{1}{x^3}$.

Solution: $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2 = 4$, therefore

$x^2 + \frac{1}{x^2} = 2$ and $x + \frac{1}{x} = 2$

$$2 \times 2 = \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + x + \frac{1}{x},$$

Hence $x^3 + \frac{1}{x^3} = 2 \times 2 - 2 = 2$

Question 7

If $x^2 + \frac{1}{x^2} = 2$ and $x > 0$, find $x^5 + \frac{1}{x^5}$.

Solution: $(x + \frac{1}{x})^2 = x^2 + \frac{1}{x^2} + 2 = 4$, therefore

$x + \frac{1}{x} = 2$, $\left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + x + \frac{1}{x}$, Hence

$$x^3 + \frac{1}{x^3} = 2 \times 2 - 2 = 2$$

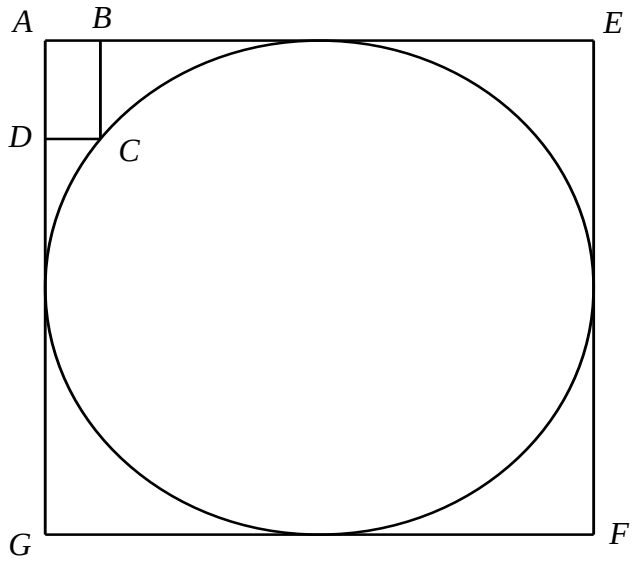
$$\left(x^4 + \frac{1}{x^4}\right)\left(x + \frac{1}{x}\right) = x^5 + \frac{1}{x^5} + x + \frac{1}{x}x^4 + \frac{1}{x^4} = (x^2 + \frac{1}{x^2})^2 - 2 = 2$$

$$\left(x^4 + \frac{1}{x^4}\right)\left(x + \frac{1}{x}\right) = x^5 + \frac{1}{x^5} + x + \frac{1}{x}$$

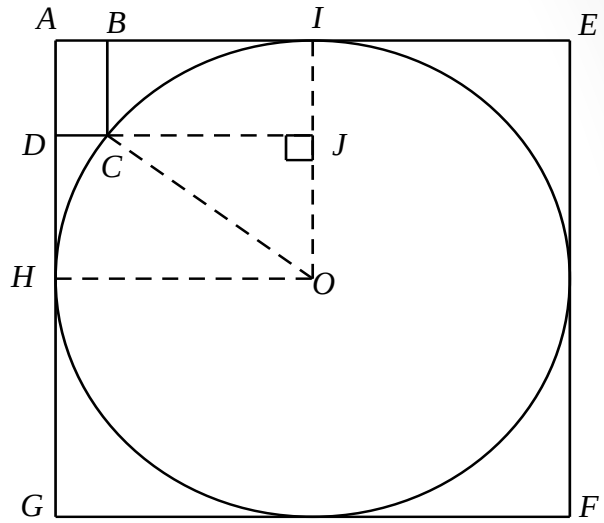
$$x^5 + \frac{1}{x^5} = 2 \times 2 - 2 = 2$$

Question 8

The diagram shows a square *AEFG* with an inscribed circle. *ABCD* is a rectangle such that *AB* = 3 cm and *AD* = 6 cm. Find the radius of the circle.



Question 8



Solution

Draw CJ and OH parallel to AE, and OI parallel to AG. Then ΔOCJ is a right-angled triangle

Let the radius of the circle be r cm. Then $OC = OH = OI = r$.

So $CJ = r - 3$ and $OJ = r - 6$.

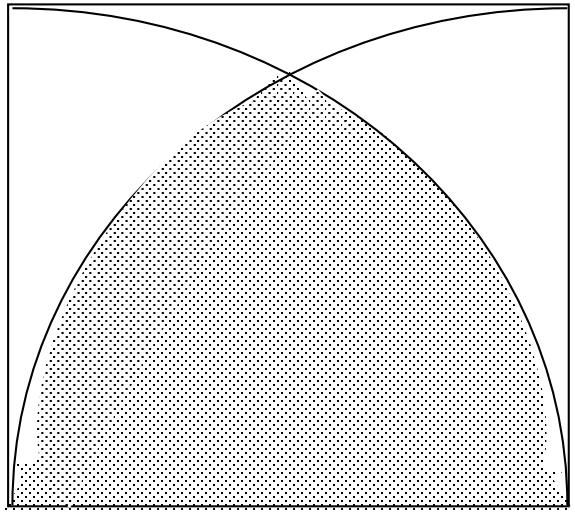
By Pythagoras' Theorem, $OC^2 = CJ^2 + OJ^2$, i.e. $r^2 = (r - 3)^2 + (r - 6)^2$.

Simplifying, we have $r^2 - 18r + 45 = 0$, i.e. $(r - 3)(r - 15) = 0$.

Since $r > 6$, then the radius of the circle is **15 cm**.

Question 9

The diagram shows a square of length 6 cm and two quarter-circular arcs of radius 6 cm. Find the area of the shaded region, leaving your answer in terms of π and surds.



Question 9

Solution

DE = *DC* (radius of quarter-circular arc *AEC*)

CE = *DC* (radius of quarter-circular arc *BED*)

Thus, DCDE is an equilateral triangle.

Area of D*CDE* $= \frac{1}{2} r^2 \sin 60°$

$= \frac{1}{2} \times 6^2 \times \frac{\sqrt{3}}{2}$

$= 9\sqrt{3}$

Area of sector *CDFE* $= \frac{60°}{360°} \times \pi r^2$

$= \frac{1}{6} \times p \times 6^2$

$= 6\pi$

Question 9

Solution

Area of segment *DFE* = area of sector *CDFE* – area of D*CDE*

$= 6\pi - 9\sqrt{3}$

Thus, area of shaded region = 2 × area of segment *DFE* + area of D*CDE*

$= 2(6\pi - 9\sqrt{3}) + 9\sqrt{3}$

$= \mathbf{12\pi - 9\sqrt{3}\ cm^2}$

Question 10

Find the next term of the following sequence:
1, 4, 9, 7, 7, 9, 4, 1, 9, 1, …

..

Question 10

Solution

Observe that the first 3 terms are perfect squares: 1^2, 2^2 and 3^2.

But subsequent terms are not perfect squares.

However, if you compare the given sequence with perfect squares (1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …), you will observe that the *n*-th term can be found by squaring *n* and then adding the digits *continuously* until a single-digit number is obtained.

$\therefore$ the next term, which is the 11^{th} term, is obtained by: $11^2 = 121$

$\rightarrow 1 + 2 + 1 =$ **4**.

Question 11

1. For each of 8 colors, I have one shirt and one tie of that color. How many shirt-and-tie outfits can I make if I refuse to wear a shirt and a tie of the same color?

2. How many license plates consist of 2 letters followed by 3 numbers?

3. How many license plates consist of 3 letters, followed by 2 even digits, followed by 2 odd digits?

4. In how many ways can I stack 5 books on a shelf?

5. Suppose that I have 6 different books, 2 of which are math books. In how many ways can I stack my 6 books on a shelf if I want a math book on both ends of the stack? **Hints:** 1

6. There are 8 sprinters in the Olympic 100-meter finals. The gold medal goes to first place, silver to second, and bronze to third. In how many ways can the medals be awarded?

7. My club has 15 members. In how many ways can we choose a president, vice-president, secretary, and treasurer, if no member can hold more than one office?

Question 11

1. $8 \times 7 = 56$
2. $26 \times 26 \times 10 \times 10 \times 10$
3. $26 \times 26 \times 5 \times 5 \times 5 \times 5$
4. $5 \times 4 \times 3 \times 2 \times 1$
5. $4 \times 3 \times 2 \times 1 \times 1$
6. $8 \times 7 \times 6$
7. $15 \times 14 \times 13 \times 12$

Question 12

A drawer contains 10 pairs of socks. Each pair is either black or white. What is the minimum number of socks that must be drawn at random from the drawer to ensure that 3 pair of socks of the same colour is selected?

Solution: 11 socks.

Considering the worst case, we draw 5 socks of each colour. We need to draw one more sock to meet the requirement. So the answer is $2 \times 5 + 1 = \mathbf{11}$.

Question 13

A drawer contains 60 pairs of socks. Each pair is one of four colours. What is the minimum number of socks that must be drawn at random from the drawer to ensure that a pair of socks of the same colour is selected?

Solution: 5 socks

Method 1: Considering the worst case, we would draw four socks, one from each colour and we would not have a pair yet. If we draw one more sock, that one will pair with one of the four socks we had drawn before.

Method 2: We have m colours and we want to find N, the minimum number of socks that must be drawn at random from the drawer to ensure that n pairs of socks of the same colour are selected: $N = 2n + m - 1$

So $N = 2n + m - 1 = 2 \times 1 + 4 - 1 = \mathbf{5}$.

Question 14

How many numbers must be selected from 15 numbers 1, 2, 3, …, 15 to ensure that there must have two numbers with the sum of 16?

Solution:

We construct 8 boxes the following way: {1, 15}, {2, 14}, {3, 13}, {4, 12}, {5, 11}, {6, 10}, {7, 9}, and {8}.

We first take out 8 numbers and one from each box. When we take the 9th number, we must take a number that is the first 7 boxes. Then we know that we have taken two numbers from the same box, and their sum must be **16**.

Question 15

Frank knows 5 women: Amy, Betty, Cheryl, Doris and Elaine.

a. 3 women are under 30 and the other 2 women are over 30.

b. 3 women are nurses and the other 2 women are teachers.

c. Amy and Cheryl are in the same age bracket.

d. Doris and Elaine are in different age brackets.

e. Betty and Elaine have the same occupation.

f. Cheryl and Doris have different occupations.

g. Of the 5 women, Frank will marry the teacher over 30.

Who will Frank marry?

Solution

Conditions *a*, *c* and *d* $\Rightarrow$ Amy and Cheryl are under 30, and Betty is over 30.
Conditions *b*, *e* and *f* $\Rightarrow$ Betty and Elaine are nurses, and Amy is a teacher.

3 women under 30	2 women over 30	3 Nurses	2 Teachers
Amy	Betty	Betty	Amy
Cheryl	?	Elaine	?
?		?	

So Amy is a teacher under 30.
Conditions *b* and *g* $\Rightarrow$ the other teacher must be over 30.
Condition *f* $\Rightarrow$ Cheryl or Doris is the other teacher over 30.
But Cheryl is under 30, so Doris is the other teacher over 30.
Frank will marry **Doris**.

Question 16

In the laboratory, three students guessed the composition of an unknown gas, every student described the gas in two sentences:
Student A said, “It is not oxygen. It is not hydrogen. ”
Student B said, “It is not nitrogen. It is carbon dioxide. ”
Student C said, “It is not carbon dioxide. It is oxygen. ”
After testing the gas, it was discovered that one of the students said two correct sentences, one said only one correct sentence, and one said the two sentences wrongly. What was the gas actually?

Solution

Student A can say either two correct sentences or only one correct sentence. Student A cannot said two sentences wrongly since the unknown gas cannot be both oxygen and hydrogen.
If Student C said two sentences wrongly, the unknown gas will then be carbon dioxide.
In this case, Student A must say both sentences correctly. Student B must say the second sentence correctly and the first sentence wrongly. However, this will create a contradiction since the unknown gas cannot be both nitrogen and carbon dioxide.
If Student B said two sentences wrongly, the unknown gas will then be nitrogen.
In this case, Student A must say both sentences correctly. Student C must say the first sentence correctly and the second sentence wrongly. This case is possible.
So, the unknown gas is nitrogen.